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THEORETICAL INVESTIGATION OF TRANSONIC SIMILARITY FOR BODIES OF REVOLUTION

By W. Perl and Milton M. Klein

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

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SUMMARY

A solution for the compressible potential flow past slender bodies of revolution has been derived by an iteration procedure similar to that of the Rayleigh-Janzen and Prandtl-Ackeret methods. The solution has been analyzed with respect to transonic similarity. The results obtained are in approximate agreement with those of von Kármán in the region of the flow field not too close to the body. In the neighborhood of the body, a different similarity law is obtained. This new similarity law holds for variations in thickness ratio and Mach number, but not for variations in specific-heat ratio. In addition, this law appears to be limited in applicability to extremely slender bodies of revolution probably outside the range of practical interest. The differences between the results of the present investigation and those of von Kármán are interpreted in terms of the manner in which the boundary condition on the body is satisfied and of the nature of the singularity of the solution near the axis.

INTRODUCTION

Transonic similarity rules for thin airfoils and slender bodies of revolution have been derived by von Kármán (reference 1). These rules for the case of two-dimensional flow are verified in reference 2 by an iteration procedure similar to that of the Rayleigh-Janzen method (reference 3) and the Prandtl-Ackeret method (reference 4).

The analogous investigation for bodies of revolution was made at the NACA Lewis laboratory and is presented herein. A solution for the compressible potential flow past a slender body of revolution is obtained by the same iteration and transonic limiting procedure used in reference 2. As in reference 2, in each step of the iteration procedure the boundary conditions were satisfied on the body.

The solution obtained yielded transonic similarity rules that are, in part, different from those of reference 1. The differences appear to result from the method of satisfying the boundary condition on the body and from the nature of the singularity of the solution near the axis.

As in the two-dimensional investigation, the influence of stagnation points has not been considered herein.

ANALYSIS

General formulation. - The partial differential equation for a compressible, axially symmetric, isentropic and irrotational flow with free-stream velocity U in cylindrical coordinates x,r,θ (fig. 1) is

$$\left[a^2 - (U+u)^2\right]\varphi_{xx} + (a^2-v^2)\varphi_{rr} - 2(U+u)\varphi_r\varphi_{xr} + a^2\frac{\varphi_r}{r} = 0 \qquad (1)$$

in which the following notation has been used:

a local speed of sound

$U+u$ resultant velocity in x-direction

v resultant velocity in r-direction

φ perturbation velocity potential defined by $u = \varphi_x$, $v = \varphi_r$

(Subscripts denote differentiation with respect to variable noted.) (All symbols used herein are defined in the appendix.) The local speed of sound a is related to the free-stream speed of sound a_0, the ratio of specific heats γ, and the local velocity by the Bernoulli equation

$$a^2 = a_0^2 - \frac{\gamma-1}{2}(2Uu + u^2 + v^2) \qquad (2)$$

In accordance with the Prandtl-Ackert type of procedure, equation (1) will be written in a form in which the linear terms appear on the left side of the equation and the nonlinear terms on the right side. A solution will then be sought in the range of free-stream Mach number close to 1 and thickness ratio close to 0 on the assumption that the flow pattern obtained by inclusion of the nonlinear terms will differ by only a small amount from that obtained with only the linear terms. This assumption will then be made plausible by the resulting form of the solution. The coefficient of φ_{xx} in equation (1) is therefore expressed, with the aid of equation (2), in the form

$$a^2 - (U+u)^2 = \beta^2 a^2 - a_0^2\left\{\Gamma_M\left[\frac{2u}{U} + \left(\frac{u}{U}\right)^2\right] + \frac{\gamma-1}{2}M_0^4\left(\frac{v}{U}\right)^2\right\} \qquad (3)$$

where

M_0 free-stream Mach number, U/a_0

$$\beta^2 = 1-M_0^2 \tag{4a}$$

$$\Gamma_M = M_0^2\left(1+\frac{\gamma-1}{2}M_0^2\right) \tag{4b}$$

For convenience, the free-stream velocity is taken as the unit velocity so that u/U, v/U, and φ/U may be written as u, v, and φ, respectively. The differential equation (1) can now be expressed in the form

$$\left(\beta^2\varphi_{xx}+\varphi_{rr}+\frac{\varphi_r}{r}\right)\left[1-\frac{\gamma-1}{2}M_0^2\left(2\varphi_x+\varphi_x^2+\varphi_r^2\right)\right]$$

$$=\left[\Gamma_M\left(2\varphi_x+\varphi_x^2\right)+\frac{\gamma-1}{2}M_0^4\varphi_r^2\right]\varphi_{xx}+M_0^2\varphi_r^2\varphi_{rr}+2M_0^2(1+\varphi_x)\varphi_r\varphi_{xr} \tag{5}$$

The boundary conditions of the problem are: (1) The perturbation velocities vanish at infinity, and (2) the flow follows the contour of the body. Thus, at infinity,

$$\varphi_x = \varphi_r = 0 \tag{6}$$

and on the body,

$$r_b = \tau g(x) \tag{7a}$$

$$\varphi_r = \tau(1+\varphi_x)\, g_x(x) \tag{7b}$$

where

τ lateral distance ratio of body

$g(x)$ function characterizing shape of body and of order of magnitude 1

All lengths are expressed in terms of the chord of the body as 1.

In order to obtain the Laplacian of φ on the left side of equation (5), the affine transformation

$$\omega = \beta r \tag{8}$$

is introduced. The differential equation (5) becomes

$$\Delta\varphi = \frac{\gamma-1}{2} M_0^{\ 2}\left(2\varphi_x+\varphi_x^{\ 2}+\beta^2\varphi_\omega^{\ 2}\right)\Delta\varphi + \left[\frac{\Gamma_M}{\beta^2}\left(2\varphi_x+\varphi_x^{\ 2}\right) + \frac{\gamma-1}{2} M_0^{\ 4}\varphi_\omega^{\ 2}\right]\varphi_{xx} + M_0^{\ 2}\beta^2\varphi_\omega^{\ 2}\varphi_{\omega\omega} + 2M_0^{\ 2}(1+\varphi_x)\varphi_\omega\varphi_{x\omega} \tag{9}$$

where the Laplacian of φ in cylindrical coordinates is

$$\Delta\varphi = \varphi_{xx} + \varphi_{\omega\omega} + \frac{\varphi_\omega}{\omega} \tag{10}$$

The boundary conditions (7a) and (7b) become,

at infinity,

$$\varphi_x = \varphi_\omega = 0 \tag{11}$$

on the body,

$$\omega_b = \tau\beta\ g(x) \tag{12a}$$

$$\varphi_\omega - \frac{\tau}{\beta}\, g_x(x)\ \varphi_x = \frac{\tau}{\beta}\, g_x(x) \tag{12b}$$

The formulation of the problem so far is exact. A solution is now sought that is applicable in the range τ and β close to zero ($\tau \sim 0$, $\beta \sim 0$). This solution is referred to hereinafter as "the small-perturbation transonic limiting solution." In the range of τ and β under consideration ($\tau \sim 0$, $\beta \sim 0$), the perturbation velocities will be assumed small

compared with free-stream velocity or $|u| \ll 1$, $|v| \ll 1$. Thus, as is usual in this type of procedure, the right side of equation (9) is considered to produce a small perturbation from the linear case and a solution of the system of equations (9), (11), and (12) will be sought in the form

$$\varphi = \overset{1}{\varphi} + \overset{2}{\varphi} + \overset{3}{\varphi} + \ldots \tag{13}$$

in which each term is of a lesser order of magnitude than the preceding one. The following boundary conditions on $\overset{1}{\varphi}, \overset{2}{\varphi}, \ldots$ will accordingly be taken as the equivalent of the boundary conditions (11) and (12):

At infinity,

$$\overset{n}{\varphi}_x = \overset{n}{\varphi}_\omega = 0; \qquad n = 1,2,3, \ldots \tag{14}$$

On the body,

$$\omega_b = \beta\tau\, g(x) \tag{15a}$$

$$\overset{1}{\varphi}_\omega - \frac{\tau}{\beta} g_x(x)\, \overset{1}{\varphi}_x = \frac{\tau}{\beta} g_x(x) \tag{15b}$$

$$\overset{n}{\varphi}_\omega - \frac{\tau}{\beta} g_x(x)\, \overset{n}{\varphi}_x = 0; \qquad n \neq 1 \tag{15c}$$

In order to obtain a solution of the system of equations (9), (13), (14), and (15), equation (13) is inserted into the differential equation (9) and a typical Laplacian term on the left, such as $\Delta\overset{n}{\varphi}$, is equated to the sum of those terms on the right that contain the superscript n-1 and that may also contain any of the superscripts n-2, n-3, . . . 1. The right side of the differential equation for $\overset{n}{\varphi}$ consists of a sum of terms of which, for a range of M_0 near 1 and τ near zero, some will be of highest order of magnitude. The solution corresponding to these terms constitutes the small-perturbation transonic limiting solution for $\overset{n}{\varphi}$.

<u>Transformation to prolate-elliptic coordinates</u>. - For the problem of the flow past an isolated body of revolution, it is convenient in satisfying the boundary conditions to use a system of prolate-elliptic

coordinates (reference 5). The transformation from cylindrical coordinates x,ω to elliptic coordinates μ,λ is given by

$$x = \mu\lambda \tag{16a}$$

$$\omega = \sqrt{(\lambda^2-1)(1-\mu^2)} \tag{16b}$$

The surfaces λ = constant, μ = constant are confocal ellipsoids and hyperboloids of two sheets, respectively, the common foci being at $x = \pm 1$, $\omega = 0$. The values of λ may range from 1 to infinity, whereas μ varies between -1 and +1.

The Laplacian of φ is, in elliptic coordinates,

$$\Delta\varphi = \frac{1}{(\lambda^2-\mu^2)}\left\{\left[(1-\mu^2)\varphi_\mu\right]_\mu + \left[(\lambda^2-1)\varphi_\lambda\right]_\lambda\right\} \tag{17}$$

Transformations from derivatives with respect to x and ω to derivatives with respect to μ and λ will be needed for subsequent analysis. From the transformation equation (16), these relations are

$$\varphi_x = \frac{\mu(\lambda^2-1)}{(\lambda^2-\mu^2)}\varphi_\lambda + \frac{\lambda(1-\mu^2)}{(\lambda^2-\mu^2)}\varphi_\mu \tag{18}$$

$$\varphi_\omega = \frac{\sqrt{(\lambda^2-1)(1-\mu^2)}}{(\lambda^2-\mu^2)}(\lambda\varphi_\lambda-\mu\varphi_\mu) \tag{19}$$

$$\varphi_{xx} = \frac{(\lambda^2-1)(1-\mu^2)}{(\lambda^2-\mu^2)^3}\left[\lambda(\lambda^2+3\mu^2)\varphi_\lambda-\mu(\mu^2+3\lambda^2)\varphi_\mu\right] + \frac{\mu^2(\lambda^2-1)^2}{(\lambda^2-\mu^2)^2}\varphi_{\lambda\lambda} +$$

$$\frac{2\mu\lambda(1-\mu^2)(\lambda^2-1)}{(\lambda^2-\mu^2)^2}\varphi_{\mu\lambda} + \frac{\lambda^2(1-\mu^2)^2}{(\lambda^2-\mu^2)^2}\varphi_{\mu\mu} \tag{20}$$

$$\varphi_{\omega\omega} = \frac{1}{(\lambda^2-\mu^2)^3} \left\{ \left[(\lambda^2-2\mu^2\lambda^2+\mu^2)(1-\mu^2)-\mu^2(\lambda^2-1)(2-\lambda^2-\mu^2) \right] \lambda\varphi_\lambda + \right.$$

$$\left. \left[(\lambda^2-2\mu^2\lambda^2+\mu^2)(\lambda^2-1)-\lambda^2(1-\mu^2)(2-\lambda^2-\mu^2) \right] \mu\varphi_\mu \right\} +$$

$$\frac{(\lambda^2-1)(1-\mu^2)}{(\lambda^2-\mu^2)^2} \left(\lambda^2\varphi_{\lambda\lambda} + \mu^2\varphi_{\mu\mu} - 2\mu\lambda\varphi_{\mu\lambda} \right) \tag{21}$$

$$\varphi_{x\omega} = \frac{\sqrt{(\lambda^2-1)(1-\mu^2)}}{(\lambda^2-\mu^2)^2} \left\{ \mu \left[\frac{\lambda^2(3-\lambda^2) + \mu^2(1-3\lambda^2)}{(\lambda^2-\mu^2)} \right] \varphi_\lambda - \right.$$

$$\lambda \left[\frac{\mu^2(3-\mu^2) + \lambda^2(1-3\mu^2)}{(\lambda^2-\mu^2)} \right] \varphi_\mu +$$

$$\left. \mu\lambda(\lambda^2-1)\varphi_{\lambda\lambda} + (\lambda^2-2\lambda^2\mu^2+\mu^2)\varphi_{\mu\lambda} - \mu\lambda(1-\mu^2)\varphi_{\mu\mu} \right\} \tag{22}$$

The variable λ has the limiting value 1 on a body of revolution as the thickness ratio τ of the body approaches zero. It will therefore be convenient to have the limiting forms, for $\lambda \rightarrow 1$, of equations (18) to (22). Defining a new variable l by

$$l = \lambda^2-1 \tag{23}$$

and denoting by $\sim$ the limiting form for $\lambda \rightarrow 1$, equations (18) to (22) take on the limiting forms

$$\varphi_x \sim \frac{2\mu}{1-\mu^2} l\varphi_l + \varphi_\mu \tag{24}$$

$$\varphi_\omega \sim \frac{l^{1/2}}{(1-\mu^2)^{1/2}} (2\varphi_l - \mu\varphi_\mu) \tag{25}$$

$$\varphi_{xx} \sim \frac{1}{(1-\mu^2)^2} \Big[2(1+3\mu^2) l\varphi_l - \mu(3+\mu^2) l\varphi_\mu + 4\mu^2 l^2 \varphi_{ll} + 4\mu(1-\mu^2) l\varphi_{l\mu} + (1-\mu^2)^2 \varphi_{\mu\mu}\Big] \tag{26}$$

$$\varphi_{\omega\omega} \sim \frac{1}{(1-\mu^2)} \left[2\varphi_l - \mu\varphi_\mu + l\left(4\varphi_{ll} - 4\mu\varphi_{l\mu} + \mu^2\varphi_{\mu\mu}\right)\right] \tag{27}$$

$$\varphi_{x\omega} \sim \frac{l^{1/2}}{(1-\mu^2)^{3/2}} \Big[4\mu\varphi_l - (1+\mu^2)\varphi_\mu + 4\mu l\varphi_{ll} + 2(1-\mu^2)\varphi_{l\mu} - \mu(1-\mu^2)\varphi_{\mu\mu}\Big] \tag{28}$$

The boundary condition (14) becomes, in elliptic coordinates, at infinity,

$$\left.\begin{aligned} \mu \frac{(\lambda^2-1)}{(\lambda^2-\mu^2)} \overset{n}{\varphi}_\lambda + \frac{\lambda(1-\mu^2)}{(\lambda^2-\mu^2)} \overset{n}{\varphi}_\mu &= 0 \\ \frac{\sqrt{(\lambda^2-1)(1-\mu^2)}}{(\lambda^2-\mu^2)} (\lambda\overset{n}{\varphi}_\lambda - \mu\overset{n}{\varphi}_\mu) &= 0 \end{aligned}\right\} n = 1,2,3,\ldots \tag{29}$$

The boundary condition (15) has, as a limiting form in elliptic coordinates, on the body,

$$l_b = \frac{\tau^2\beta^2 g^2}{(1-\mu^2)} \tag{30a}$$

$$\frac{l^{1/2}}{(1-\mu^2)^{1/2}}\left(2\overset{1}{\varphi}_l-\mu\overset{1}{\varphi}_\mu\right)-\frac{T}{\beta}\,g_x(x)\left(\frac{2\mu l}{1-\mu^2}\overset{1}{\varphi}_l+\overset{1}{\varphi}_\mu\right)=\frac{T}{\beta}\,g_x(x) \tag{30b}$$

$$\frac{l^{1/2}}{(1-\mu^2)^{1/2}}\left(2\overset{n}{\varphi}_l-\mu\overset{n}{\varphi}_\mu\right)-\frac{T}{\beta}\,g_x(x)\left(\frac{2\mu l}{1-\mu^2}\overset{n}{\varphi}_l+\overset{n}{\varphi}_\mu\right)=0;\quad n\neq 1 \tag{30c}$$

First approximation. - The solution for the first approximation $\overset{1}{\varphi}$ satisfies the homogeneous Laplace equation

$$\Delta\overset{1}{\varphi}=\left[(1-\mu^2)\overset{1}{\varphi}_\mu\right]_\mu+\left[(\lambda^2-1)\overset{1}{\varphi}_\lambda\right]_\lambda=0 \tag{31}$$

subject to the boundary conditions (29), (30a), and (30b). A solution of equation (31) appropriate to the present problem is given by (reference 5)

$$\overset{1}{\varphi}(\mu,\lambda)=\sum_{n=0}^{\infty}\overset{1}{A}_n\,Q_n(\lambda)\,P_n(\mu) \tag{32}$$

where $P_n(\mu)$ and $Q_n(\lambda)$ are the Legendre functions of the first and second kind, respectively, and the A_n are constants. The function $P_n(\mu)$ is a polynomial in μ of degree n, free of singularities for finite values of μ and is given by (reference 6)

$$P_0(\mu)=1$$

$$P_1(\mu)=\mu$$

$$P_2(\mu)=\frac{1}{2}(3\mu^2-1)$$

$$P_n(\mu)=\frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n=\sum_{j=0}^{m}\frac{(-1)^j(2n-2j)!}{2^n j!(n-j)!(n-2j)!}\mu^{n-2j} \tag{33}$$

where $m = n/2$ for n even and $m = (n-1)/2$ for n odd. The function $Q_n(\lambda)$ has a logarithmic singularity at $\lambda = \pm 1$ and is given by (reference 7)

$$Q_0(\lambda) = \frac{1}{2} \log \frac{(\lambda+1)^2}{\lambda^2-1}$$

$$Q_1(\lambda) = \frac{P_1(\lambda)}{2} \log \frac{(\lambda+1)^2}{\lambda^2-1} - 1$$

$$Q_2(\lambda) = \frac{1}{2} P_2(\lambda) \log \frac{(\lambda+1)^2}{\lambda^2-1} - \frac{3}{2} P_1(\lambda)$$

$$\begin{aligned} Q_n(\lambda) &= P_n(\lambda)\int_\lambda^\infty \frac{d\lambda}{(\lambda^2-1)\left[P_n(\lambda)\right]^2} \\ &= \frac{1}{2} P_n(\lambda) \log \frac{(\lambda+1)^2}{\lambda^2-1} - \frac{2n-1}{n-1} P_{n-1}(\lambda) - \frac{2n-5}{3(n-1)} P_{n-3}(\lambda) - \dots \end{aligned} \tag{34}$$

For large values of λ, the function $Q_n(\lambda)$ has, as limiting form, (reference 6)

$$Q_n(\lambda) \longrightarrow \frac{2^n(n!)^2}{(2n+1)!\lambda^{n+1}}$$

The function $Q_n(\lambda)$ and its derivatives therefore approach zero as λ becomes infinite, so that the boundary condition at infinity (equation (29)) is satisfied. The constants $\frac{1}{A_n}$ in equation (32) are determined so that the boundary condition at the body (equations (30a) and (30b)) is satisfied. In the neighborhood of slender bodies, λ approaches 1 so that the limiting forms of $Q_n(\lambda)$ and its derivatives are applicable near the body. These are, from equations (34),

$$Q_n(l) \sim -\frac{1}{2} \log l \tag{35a}$$

$$Q_n'(l) \sim -\frac{1}{2l} \tag{35b}$$

$$Q_n''(l) \sim \frac{1}{2l^2} \tag{35c}$$

where the primes are used to denote differentiation with respect to l. Primes will be used, where convenient, to denote differentiation when only one variable is involved. The limiting form of the function $Q_n(l)$ is evidently independent of the subscript n, so that equation (32) has a limiting form in the neighborhood of the body

$$\overset{1}{\varphi}(\mu,l) = -\frac{1}{2}\log l \sum_{n=0}^{\infty} \overset{1}{A}_n P_n(\mu) \tag{36}$$

Insertion of equation (36) into the body boundary condition (30b) for the determination of $\overset{1}{A}_n$ yields the equation

$$\frac{l_b^{1/2}}{(1-\mu^2)^{1/2}}\left[-\frac{1}{l_b}\sum_{n=0}^{\infty}\overset{1}{A}_n \overset{①}{P_n}(\mu) + \frac{\mu}{2}\log l_b \sum_{n=0}^{\infty} \overset{1}{A}_n \overset{②}{P_n}'(\mu)\right] +$$

$$\frac{\tau}{\beta} g_x(x)\left[\frac{\mu}{1-\mu^2}\sum_{n=0}^{\infty}\overset{1}{A}_n \overset{③}{P_n}(\mu) + \frac{1}{2}\log l_b \sum_{n=0}^{\infty} \overset{1}{A}_n \overset{④}{P_n}'(\mu)\right] = \frac{\tau}{\beta} g_x(x) \tag{37}$$

For convenience in discussion, identifying supernumerals have been placed above the various terms in equation (37). These terms will next be compared in the small-perturbation transonic limit. The ratio of term ② to term ① is, by equation (30a), of order $\tau^2\beta^2 \log \tau\beta$, whereas the ratio of term ③ to term ④ is of order $1/\log \tau\beta$. Both of these ratios have a limiting value of zero regardless of how τ and β individually approach zero. It is therefore permissible to neglect ② with respect to ① and ③ with respect to ④. The ratio of term ④ to term ① is of order $\tau^2 \log \tau\beta$; the limiting value of this ratio

depends on the manner in which τ and β approach zero. Although $\tau^2 \log \tau\beta$ will, in general, be small for small τ, values of β exist that, for small but fixed values of τ, will make $\tau^2 \log \tau\beta$ comparable to 1. The quantities τ and β are to be regarded as independent variables in going to the small-perturbation transonic limit; it is therefore necessary to retain term 4 along with term 1 in equation (37). Term 4 arises from the φ_x term in the boundary condition (12) and is generally neglected in small-perturbation analysis. When terms 2 and 3 are neglected, and $\log\left[g/(1-\mu^2)^{1/2}\right]$ is neglected relative to $\log \tau\beta$, equation (37) reduces to

$$\sum_{n=0}^{\infty} \overset{1}{A}_n P_n(\mu) - \kappa g g_x \sum_{n=0}^{\infty} \overset{1}{A}_n P_n'(\mu) = -\tau^2 g g_x \tag{38}$$

where

$$\kappa = \tau^2 \log \tau\beta \tag{39}$$

Both sides of equation (38) may be expressed as series of orthogonal Legendre polynomials in μ and the coefficients of corresponding Legendre polynomials equated to each other. The result is a set of simultaneous linear equations for the $\overset{1}{A}_n$, the solution of which may be indicated as

$$\overset{1}{A}_n = \tau^2 \overset{1}{a}_n(\kappa,b) \tag{40}$$

where $\overset{1}{a}_n(\kappa,b)$ denotes a function of the parameter κ and of body shape; the functions g and g_x are of order of magnitude 1 so that it is plausible to assume $\overset{1}{a}_n$ is also of order 1.

The solution for the first approximation $\overset{1}{\varphi}$ is then

$$\overset{1}{\varphi} = \tau^2 \sum_{n=0}^{\infty} \overset{1}{a}_n Q_n(\lambda) P_n(\mu) \tag{41}$$

Near the body $\overset{1}{\varphi}$ has the form (from equation (35a))

$$\overset{1}{\varphi} \sim -\tau^2 \frac{\log l}{2} \sum_{n=0}^{\infty} \overset{1}{a}_n P_n(\mu) \tag{42}$$

The perturbation velocity $\overset{1}{\varphi}_x$ near the body is, from equations (24) and (42),

$$\overset{1}{\varphi}_x \sim -\tau^2 \left[\frac{\mu}{1-\mu^2} \sum_{n=0}^{\infty} \overset{1}{a}_n P_n(\mu) + \frac{1}{2} \log l \sum_{n=0}^{\infty} \overset{1}{a}_n P_n{}'(\mu) \right] \tag{43}$$

On the body or in the flow field not too far from the body, the second term in the bracket dominates the first term in the transonic range $\beta \sim 0$. In particular, on the body φ_x has the form

$$\overset{1}{\varphi}_x \sim -\tau^2 \log \tau\beta \sum_{n=0}^{\infty} \overset{1}{a}_n P_n{}'(\mu) = -\kappa \sum_{n=0}^{\infty} \overset{1}{a}_n P_n{}'(\mu) \tag{44}$$

The radial velocity component $\overset{1}{\varphi}_r$ near the body is, from equations (8) and (25),

$$\overset{1}{\varphi}_r = \beta \overset{1}{\varphi}_\omega \sim \frac{\tau^2 \beta l^{1/2}}{(1-\mu^2)^{1/2}} \left[-\frac{1}{l} \sum_{n=0}^{\infty} \overset{1}{a}_n P_n(\mu) + \frac{\mu}{2} \log l \sum_{n=0}^{\infty} \overset{1}{a}_n P_n{}'(\mu) \right] \tag{45}$$

The second term in the bracket is negligible compared with the first term, so that on the body equation (45) assumes the form

$$\overset{1}{\varphi}_r \sim -\frac{\tau}{g} \sum_{n=0}^{\infty} \overset{1}{a}_n P_n(\mu) \tag{46}$$

In the flow field, which will be defined as the region for which l and $\log l$ are considered to be of order 1, the distrubance velocity $\overset{1}{\varphi}_x$ is obtained from equations (18) and (41), whereas the radial velocity $\overset{1}{\varphi}_r$ is given by equations (8), (19), and (41). Because only the order of magnitude of the solution in the flow field, rather than its specific form, is desired, the process may be considerably simplified. The

functions $Q_n(\lambda)$ and $P_n(\mu)$ will contribute only terms of order of magnitude 1 to the solution. The form of the solution in the flow field may therefore be obtained from the solution near the body by taking l and $\log l$ of order 1. The solution for φ_x is therefore, from equation (43), of the form

$$\overset{1}{\varphi}_x \sim \tau^2 g_1(\kappa, b) \tag{47}$$

where $g_1(\kappa,b)$ is a function of κ and of body shape, and is of order of magnitude 1. The radial velocity $\overset{1}{\varphi}_r$ in the flow field has the form, from equation (45),

$$\overset{1}{\varphi}_r \sim \tau^2\beta\, h_1(\kappa, b) \tag{48}$$

where $h_1(\kappa,b)$ is a function of κ and of body shape, and is of order of magnitude 1.

The resultant velocity increment $\overset{1}{\Lambda}$ is defined by

$$\overset{1}{\Lambda} = \left[(1+\overset{1}{\varphi}_x)^2 + (\overset{1}{\varphi}_r)^2\right]^{1/2} - 1 \tag{49}$$

The ratio $(\overset{1}{\varphi}_r)^2/\overset{1}{\varphi}_x$ is of order $\tau^2\beta^2$ in the flow field and of order $1/\log \tau\beta$ on the body so that $(\overset{1}{\varphi}_r)^2$ may be neglected. The resultant velocity increment $\overset{1}{\Lambda}$ is therefore approximated by $\overset{1}{\varphi}_x$.

<u>Second approximation</u>. - The second approximation $\overset{2}{\varphi}$ is a solution of the non-homogeneous Laplace equation

$$\Delta\overset{2}{\varphi} = \frac{\gamma-1}{2} M_0^2 \overset{\textcircled{1}}{(2\overset{1}{\varphi}_x + \overset{1}{\varphi}_x{}^2 + \beta^2 \overset{1}{\varphi}_\omega{}^2)}\Delta\overset{1}{\varphi} + \left[\frac{\Gamma_M}{\beta^2}\overset{\textcircled{2}}{(2\overset{1}{\varphi}_x + \overset{1}{\varphi}_x{}^2)} + \overset{\textcircled{3}}{\frac{\gamma-1}{2} M_0^4 \overset{1}{\varphi}_\omega{}^2}\right]\overset{1}{\varphi}_{xx} +$$

$$\overset{\textcircled{4}}{M_0^2\beta^2\overset{1}{\varphi}_\omega{}^2\overset{1}{\varphi}_{\omega\omega}} + 2M_0^2(\overset{\textcircled{5}\,\textcircled{6}}{1+\overset{1}{\varphi}_x})\overset{1}{\varphi}_\omega\overset{1}{\varphi}_{x\omega} \tag{50}$$

which satisfies the boundary conditions (29), (30a), and (30c). In solving equation (50), only those terms on the right side that are of highest order of magnitude in the small-perturbation transonic limit $\tau \longrightarrow 0$, $\beta \longrightarrow 0$ will be retained. The plausible assumption is made in this connection that the highest-order part of the solution of a complete differential equation is the same as the complete solution of the differential equation in which only highest-order terms have been retained. For further simplification, an explicit solution will be sought only near and on the body, so that the limiting forms for the various derivatives in equations (24) to (28) may be used. The subsequent analysis will therefore only indicate implicitly the form of the solution in the flow field.

The highest-order terms in $\overset{1}{\varphi}_x$ and $\overset{1}{\varphi}_\omega$ are, from equations (43) and (45),

$$\overset{1}{\varphi}_x \sim -\tau^2 \frac{R'}{2} \log l$$

$$\overset{1}{\varphi}_\omega \sim -\frac{\tau^2}{l^{1/2}} \frac{R}{(1-\mu^2)^{1/2}}$$

where $R = R(\kappa, b, \mu) = \sum_{n=0}^{\infty} \overset{1}{a}_n(\kappa, b)\, P_n(\mu)$ and the primes denote differentiation with respect to μ. The highest-order contributions to the second-order derivatives of $\overset{1}{\varphi}$ are, by equations (41) and (24) to (28),

$$\overset{1}{\varphi}_{xx} \sim -\tau^2 \frac{R''}{2} \log l$$

$$\overset{1}{\varphi}_{x\omega} \sim -\frac{\tau^2}{l^{1/2}} \frac{R'}{(1-\mu^2)^{1/2}}$$

$$\overset{1}{\varphi}_{\omega\omega} \sim \frac{\tau^2}{l} \frac{R}{(1-\mu^2)}$$

The various terms on the right side of equation (50) will now be examined with regard to their orders of magnitude. Term 1 vanishes

by virtue of the solution for the first approximation. The ratio of term (3) to term (2) is of order $\tau^2\beta^2/l \log l$ and approaches zero both in the flow field and on the body. It is therefore permissible to neglect term (3). The ratio of term (4) or (5) to term (3) is of order $\beta^2/l \log l$, which is negligible in the flow field; near the body, the ratio may be small or large depending on the manner in which $\tau \longrightarrow 0$ and $\beta \longrightarrow 0$. Terms (4) and (5) must therefore be retained and equation (50) for the second approximation $\overset{2}{\varphi}$ reduces to

$$\frac{\Delta\overset{2}{\varphi}}{\tau^4} = \frac{\Gamma_M}{\beta^2}\left(\frac{1}{2} R'R'' \log^2 l - \frac{1}{8}(R')^2R''\tau^2\log^3 l\right) + \frac{M_0{}^2R^3}{(1-\mu^2)^2}\frac{\tau^2\beta^2}{l^2} +$$

$$2M_0{}^2\frac{RR'}{(1-\mu^2)}\frac{1}{l} - M_0{}^2\frac{R(R')^2}{(1-\mu^2)}\frac{\tau^2 \log l}{l} \tag{51}$$

A typical term on the right side of equation (51) is, aside from factors depending on the physical parameters τ, γ, and M_0, the limiting form for $\lambda \longrightarrow 1$ of some function of μ and λ, which may be denoted by $f(\mu,\lambda)$. In order to solve equation (51), it is assumed that the product of $f(\mu,\lambda)$ and $(\lambda^2-\mu^2)$ may be expanded into a series of orthogonal Legendre polynomials. The right side of equation (51) may contain singularities at the end points $\mu = \pm 1$ because of the factor $\frac{1}{1-\mu^2}$, which becomes infinite at these points. It may therefore be necessary to exclude small but finite neighborhoods about $\mu = \pm 1$ from the interval of expansion and consequently from the domain of definition of $\overset{2}{\varphi}$. At this point the assumption is made that the effect of the stagnation points, if they occur, is negligible. The expansion for $f(\mu,\lambda)$ may then be expressed by

$$(\lambda^2-\mu^2)\ f(\mu,\lambda) = \sum_{n=0}^{\infty} \alpha_n\ h_n(\lambda)\ P_n(\mu) \tag{52}$$

where α_n depends on body shape and h_n is a function of λ. Then the solution of equation (51) corresponding to the term $f(\mu,\lambda)$ is, from equation (17), the solution of the non-homogeneous equation

$$(\lambda^2-\mu^2)\Delta\varphi = \left[(1-\mu^2)\varphi_\mu\right]_\mu + \left[(\lambda^2-1)\varphi_\lambda\right]_\lambda = \tau^4\sum_{n=0}^{\infty}\alpha_n\, h_n(\lambda)\, P_n(\mu) \tag{53}$$

A solution of equation (53) is now sought in the form

$$\varphi = \tau^4\sum_{n=0}^{\infty}\alpha_n\, q_n(\lambda)\, P_n(\mu) \tag{54}$$

where $q_n(\lambda)$ is a function of λ to be determined.

Inserting equation (54) in equation (53) and noting that $P_n(\mu)$ satisfies Legendre's differential equation yields, when coefficients of $P_n(\mu)$ are equated, the non-homogeneous Legendre differential equation

$$\left[(\lambda^2-1)\; q_n'(\lambda)\right]' - n(n+1)\; q_n(\lambda) = h_n(\lambda) \tag{55}$$

The solution of equation (55) is the sum of the complementary solution taken proportional to $Q_n(\lambda)$ to satisfy the boundary condition at infinity and the particular integral $r_n(\lambda)$, which may be expressed in terms of two indefinite integrals by

$$r_n(\lambda) = P_n(\lambda)\int\left[\frac{1}{(\lambda^2-1)\; P_n^2(\lambda)}\int h_n(\lambda)\; P_n(\lambda)\; d\lambda\right]d\lambda \tag{56}$$

The function r_n satisfies the boundary conditions at infinity. The solution of equation (53) may therefore be written as

$$\varphi = \tau^4\left[\sum_{n=0}^{\infty}A_n\, Q_n(\lambda)\, P_n(\mu) + \sum_{n=0}^{\infty}\alpha_n\, r_n(\lambda)\, P_n(\mu)\right] \tag{57}$$

The limiting form of equation (56) for $\lambda \longrightarrow 1$ or $l \longrightarrow 0$ is

$$r(l) \sim \frac{1}{4}\int\frac{dl}{l}\int h(l)\; dl \tag{58}$$

The subscript n is omitted from $r(l)$ inasmuch as $h_n(\lambda)$ is independent of the subscript n in the limit $l \longrightarrow 0$. Equation (57) then becomes, for $l \longrightarrow 0$,

$$\varphi \sim \tau^4 \left[-\frac{1}{2} \log l \sum_{n=0}^{\infty} A_n \, P_n(\mu) + r(l) \sum_{n=0}^{\infty} \alpha_n \, P_n(\mu) \right] \tag{59}$$

The quantity $r(l)$ may now be evaluated for each term on the right side of equation (51) and the solution for $\overset{2}{\varphi}$ consequently is

$$\frac{\overset{2}{\varphi}}{\tau^4} \sim -\frac{1}{2} \log l \sum_{n=0}^{\infty} \overset{2}{A}_n \, P_n(\mu) + \frac{1}{8} \left(1-\mu^2\right) \frac{\Gamma_M}{\beta^2} \left[R'R'' l \log^2 l - \frac{1}{4} (R')^2 R'' \tau^2 l \log^3 l \right] +$$

$$\frac{1}{4} M_0^{\,2} \frac{R^3}{\left(1-\mu^2\right)} \frac{\tau^2 \beta^2}{l} + \frac{1}{4} M_0^{\,2} RR' \log^2 l - \frac{1}{24} M_0^{\,2} R(R')^2 \tau^2 \log^3 l \tag{60}$$

Equation (60) explicitly indicates the solution only near the body, where $l \sim 0$. In the flow field ($l \sim 1$ and $\log l \sim 1$), equation (60) indicates the solution only as to order of magnitude, as previously mentioned. This limitation, however, is unimportant in the investigation. The limiting form of the solution as $\tau \sim 0$ (for example, equation (60)) is used herein because of the resultant simplification in the analysis. The constants $\overset{2}{A}_n$ in equation (60) are determined from the boundary conditions (30a) and (30c). Insertion of equation (60) into equation (30c) yields

$$
\begin{aligned}
&\frac{2l^{1/2}}{(1-\mu^2)^{1/2}}\left\{-\frac{1}{2l}\sum_{n=0}^{\infty} A_n^{2}\, P_n(\mu) + \frac{1}{8}\,(1-\mu^2)\,\frac{\Gamma_M}{\beta^2}\left[R'R''\,\log^2 l - \frac{1}{4}\,(R')^2R''\tau^2\,\log^3 l\right] - \right.\\
&\left.\quad \frac{1}{4}\,M_0^{\,2}\,\frac{R^3}{(1-\mu^2)}\,\frac{\tau^2\beta^2}{l^2} + \frac{1}{2}\,M_0^{\,2}RR'\,\frac{\log l}{l} - \frac{1}{8}\,M_0^{\,2}R(R')^2\tau^2\,\frac{\log^2 l}{l} - \frac{\mu}{2}\,\varphi_\mu^{\,2}\right\} - \\
&\frac{\tau}{\beta}\,g_x(x)\left(\frac{2\mu}{1-\mu^2}\left\{-\frac{1}{2}\sum_{n=0}^{\infty} A_n^{2}\, P_n(\mu) + \frac{1}{8}\,(1-\mu^2)\,\frac{\Gamma_M}{\beta^2}\left[R'R''l\,\log^2 l - \frac{1}{4}\,(R')^2R''\tau^2 l\,\log^3 l\right] - \right.\right.\\
&\left.\quad \frac{1}{4}\,M_0^{\,2}\,\frac{R^3}{(1-\mu^2)}\,\frac{\tau^2\beta^2}{l} + \frac{1}{2}\,M_0^{\,2}RR'\,\log l - \frac{1}{8}\,M_0^{\,2}R(R')^2\tau^2\,\log^2 l\right\} - \\
&\frac{1}{2}\,\log l\sum_{n=0}^{\infty} A_n^{2}\, P_n'(\mu) + \frac{1}{8}\,\frac{\Gamma_M}{\beta^2}\left\{\left[(1-\mu^2)R'R''\right]'\,l\,\log^2 l - \frac{1}{4}\left[(1-\mu^2)(R')^2R''\right]'\tau^2 l\,\log^3 l\right\} + \\
&\left.\quad \frac{1}{4}\,M_0^{\,2}\left(\frac{R^3}{(1-\mu^2)}\right)'\,\frac{\tau^2\beta^2}{l} + \frac{1}{4}\,M_0^{\,2}(RR')'\,\log^2 l - \frac{1}{24}\,M_0^{\,2}\left[R(R')^2\right]'\,\tau^2\,\log^3 l\right) \\
&= 0 \qquad\qquad (61)
\end{aligned}
$$

where l is to be evaluated at $l_b = \frac{\tau^2\beta^2 g^2}{1-\mu^2}$. The unchecked terms in equation (61) become negligible compared with the checked terms in the small-perturbation transonic limit. Equation (61) therefore reduces to

$$\sum_{n=0}^{\infty} \overset{2}{A}_n P_n(\mu) - \kappa g g_x \sum_{n=0}^{\infty} \overset{2}{A}_n P_n'(\mu)$$

$$= \log \tau\beta \left(2M_0{}^2 RR' + \kappa \left[\Gamma_M g^2 R'R'' - M_0{}^2 R(R')^2 - M_0{}^2 g g_x (RR')' \right] - \kappa^2 \left\{ \frac{\Gamma_M}{2} g^2 (R')^2 R'' - \frac{1}{3} M_0{}^2 g g_x \left[R(R')^2 \right] \right\} \right) \qquad (62)$$

Equation (62) may be solved for the coefficients $\overset{2}{A}_n$ in the same manner as was indicated for the coefficients $\overset{1}{A}_n$ occurring in the first approximation $\overset{1}{\varphi}$. As in equation (52), it may be necessary to exclude the region around the end points $\mu = \pm 1$. The result may be expressed as

$$\overset{2}{A}_n = \log \tau\beta \; \overset{2}{a}_n(\kappa, M_0, \Gamma_M, b)$$

where $\overset{2}{a}_n(\kappa, M_0, \Gamma_M, b)$ denotes a function of the parameters κ, M_0, Γ_M, and the body shape and is of the order of magnitude 1.

Although the term in R^3 cannot be neglected in equation (51), it can be neglected in equation (60) and in the boundary condition (equation (61)). Near the body, the solution for $\overset{2}{\varphi}$ given by equation (60) therefore becomes, when the R^3 term is neglected,

$$\frac{\overset{2}{\varphi}}{\tau^4} \sim -\frac{1}{2} \log \tau\beta \log l \sum_{n=0}^{\infty} \overset{2}{a}_n P_n(\mu) + \frac{1}{8}(1-\mu^2) \frac{\Gamma_M}{\beta^2} \left[R'R'' l \log^2 l - \frac{1}{4}(R')^2 R'' \tau^2 \log^3 l \right] + \frac{1}{4} M_0{}^2 RR' \log^2 l - \frac{1}{24} M_0{}^2 R(R')^2 \tau^2 \log^3 l \qquad (63)$$

An explicit solution for the potential in the flow field cannot be obtained from equation (63) but, as previously mentioned, the order of magnitude of the solution in this region may be inferred from equation (63) by taking l and $\log l$ of order 1. In the flow field, the disturbance velocities $\overset{2}{\varphi}_x$ and $\overset{2}{\varphi}_r$ thus have the form

$$\overset{2}{\varphi}_x \sim \tau^2 (c_1 \kappa + c_2 \epsilon) \tag{64}$$

$$\overset{2}{\varphi}_r = \beta \overset{2}{\varphi}_\omega \sim \tau^2 \beta (d_1 \kappa + d_2 \epsilon) \tag{65}$$

where $\epsilon = \dfrac{\tau^2 \Gamma_M}{\beta^2}$, c_1 and d_1 are functions of κ, M_0, Γ_M, and body shape, and c_2 and d_2 are functions of only κ and body shape. The c's and d's are of order of magnitude 1.

In the neighborhood of the body (equation (30a)), the quantity l begins to contribute to the order of magnitude of the terms in equation (63). The disturbance velocities on the body are thus given by

$$\overset{2}{\varphi}_x \sim \kappa^2 \left\{ -\sum_{n=0}^{\infty} \overset{2}{a}_n P_n'(\mu) + M_0^2 (RR')' - \frac{1}{3} M_0^2 \left[R(R')^2 \right]' \kappa \right\} \tag{66}$$

$$\overset{2}{\varphi}_r \sim -\frac{\pi}{g} \kappa \left\{ \left[\sum_{n=0}^{\infty} \overset{2}{a}_n P_n(\mu) - 2M_0^2 RR' \right] + \left[M_0^2 R(R')^2 - \Gamma_M g^2 R'R'' \right] \kappa + \frac{1}{2} \Gamma_M g^2 (R')^2 R'' \kappa^2 \right\} \tag{67}$$

Aside from the dependence of the constants $\overset{2}{a}_n$ on κ, the dominant terms contributing to $\overset{2}{\varphi}_x$ on the body (equation (66)), come from the $\overset{1}{\varphi}_\omega \overset{1}{\varphi}_{x\omega}$ and $\overset{1}{\varphi}_x \overset{1}{\varphi}_\omega \overset{1}{\varphi}_{x\omega}$ terms in the differential equation (50). The first term of $\overset{2}{\varphi}_x$ in the flow field (equation (64)) results from the $\overset{1}{\varphi}_\omega \overset{1}{\varphi}_{x\omega}$ term. The second term of equation (64) comes from the $\overset{1}{\varphi}_x \overset{1}{\varphi}_{xx}$ term in 2 in equation (50). The expressions for $\overset{2}{\varphi}_r$ on the body and in the flow field (equation (67) and (65), respectively) may be similarly analyzed.

DISCUSSION

Transonic similarity. - The results obtained thus far in the first two approximations may be summarized as follows:

On the body,

$$\varphi \sim -\kappa\left[R + (S - M_0^2\, RR')\kappa + \frac{1}{3} M_0^2\, R(R')^2\, \kappa^2\right] \tag{68}$$

$$\varphi_x \sim -\kappa\left\{R' + \left[S' - M_0^2 (RR')'\right]\kappa + \frac{1}{3} M_0^2 \left[R(R')^2\right]' \kappa^2\right\} \tag{69}$$

$$\varphi_r \sim -\frac{1}{g}\left\{R + (S - 2M_0^2 RR')\kappa + \left[M_0^2 R(R')^2 - \Gamma_M g^2 R'R''\right]\kappa^2 + \frac{1}{2}\Gamma_M g^2 (R')^2 R''\kappa^3\right\} \tag{70}$$

where

$$R = \sum_{n=0}^{\infty} \overset{1}{a}_n\, P_n(\mu)$$

$$S = \sum_{n=0}^{\infty} \overset{2}{a}_n\, P_n(\mu)$$

In the flow field $(l \sim 1, \log l \sim 1)$,

$$\varphi \sim \tau^2 (b_1 + b_2\kappa + b_3\epsilon) \tag{71}$$

$$\varphi_x \sim \tau^2 (c_1 + c_2\kappa + c_3\epsilon) \tag{72}$$

$$\varphi_r \sim \tau^2\beta (d_1 + d_2\kappa + d_3\epsilon) \tag{73}$$

where b_2, c_2, and d_2 are functions of κ, M_0, Γ_M, and body shape and the other b's, c's, and d's are functions of only κ and body shape. The b's, c's, and d's are of order of magnitude 1. It appears plausible to assume that higher approximations would not alter the results obtained thus far concerning the dependence of the potential on the parameters κ and ϵ.

The solution given by equations (68) to (73) will next be considered from the viewpoint of transonic similarity; that is, the possible dependence of the solution on less than three combinations of the physical parameters τ, M_0, and γ in the small-perturbation transonic limit $\tau \longrightarrow 0$, $\beta \longrightarrow 0$ will be investigated. In the limit $\tau \longrightarrow 0$, $\beta \longrightarrow 0$, the solution for the body (equations (68) to (70)) becomes a function of the parameter κ, the ratio of specific heats γ, and the body shape. Hence, for constant γ, a similarity rule exists on the body with respect to variations in τ and β through the similarity parameter κ. In the derivation of this result, it was necessary to neglect terms of order 1 in comparison with $\log l$; that is, in comparison with $\log \tau\beta$ on the body. The quantity l, however, must be extremely small before $\log l$ begins to dominate terms of order 1. The foregoing similarity rule may therefore possibly be limited to extremely slender bodies with thickness ratios not in the range of practical interest. If the body is not extremely slender, the potential in the neighborhood of the body depends not only on the parameter κ but also on the thickness ratio τ in a complicated manner. (See, for example, equations (41) and (63).)

For the flow-field solution, (equations (71) to (73)), in the limit $\tau \longrightarrow 0$, $\beta \longrightarrow 0$, the coefficients b_2, c_2, and d_2 become functions of κ, γ, and body shape, whereas the other b's, c's, and d's remain functions of κ and body shape. The flow-field solution thus becomes, for $\tau \longrightarrow 0$, $\beta \longrightarrow 0$, a function of three parameters κ, ϵ, and γ, so that apparently no simple similarity law exists in the flow field. The original physical parameters of the problem (τ, M_0, and γ) have not been reduced in number, so that no apparent simplification of the dependence of the potential upon the physical parameters has been achieved. Figure 2, which presents curves of $|\kappa|/\epsilon$ against β for several values of the thickness ratio τ, shows, however, that the parameter ϵ is much larger than the parameter κ for small values of β. It then appears desirable to consider those flow-field solutions in which ϵ is of order of magnitude 1 and κ is negligible compared with ϵ.

The essential modifications in the analysis previously given are that the term in φ_x in the body boundary condition (7b) may now be neglected in view of equation (38) and that only the terms in $\varphi_x\varphi_{xx}$,

$\varphi_\omega \varphi_{x\omega}$ and $\varphi_\omega^2 \varphi_{\omega\omega}$ need be considered on the right side of the differential equation (9). The term in $\varphi_x \varphi_{xx}$ is the dominant term in the flow field, whereas the $\varphi_\omega \varphi_{x\omega}$ and $\varphi_\omega^2 \varphi_{\omega\omega}$ terms are dominant in the neighborhood of the body. It is therefore necessary to retain the terms $\varphi_\omega \varphi_{x\omega}$ and $\varphi_\omega^2 \varphi_{\omega\omega}$ for the body boundary condition in order to obtain the constants A_n of the complementary solution.

By carrying through the analysis directly or by inspection of the solution equations (71) to (73), the potential in the flow field may be expressed in terms of the single parameter ϵ. Because of the simplifications resulting from the condition $\kappa << \epsilon$, this analysis for the solution in the flow field has been carried through the first three approximations. In the third approximation, as well as in the second approximation (equation (63)), the complementary solution becomes negligible compared with the particular solution corresponding to the $\varphi_x \varphi_{xx}$ term in the flow field. Thus the results for the flow-field region to three approximations may be expressed as

$$F = f_1 \epsilon + f_2 \epsilon^2 + f_3 \epsilon^3 \tag{74}$$

where

$$F = \frac{\Gamma_M}{\beta^2} \varphi \tag{75}$$

and f_1, f_2, and f_3 are functions only of body shape and are of order of magnitude 1.

In establishing equation (74), it was unnecessary to make the strong assumption $\log l >> 1$ but rather the much weaker ones, $\tau^2 << 1$, $\beta^2 << 1$ and $\log \tau\beta \approx 1$. It is therefore probable that the similarity law for the flow field given by equation (74) holds for a wider range of thickness ratios than the similarity law for the body (equation (68)). The existence of different similarity laws for the flow field and for the body would indicate a transition region where a more complicated relation obtains.

Comparison with reference 1. - The results obtained in this investigation are somewhat different from those of reference 1, where a single similarity parameter equivalent to the parameter ϵ is derived for both the flow field and the body. These differences may perhaps be understood as follows:

In reference 1, the boundary condition at the body is written in a form essentially equivalent to

$$r\varphi_r = \tau^2 gg_x \qquad \text{for} \qquad r \longrightarrow 0$$

or

$$\omega F_\omega = \epsilon gg_x \qquad \text{for} \qquad \omega \longrightarrow 0 \tag{76}$$

and nonlinear terms in the differential equation for the potential other than the $\varphi_x\varphi_{xx}$ term are neglected. When only the $\varphi_x\varphi_{xx}$ term on the right side is retained and the transformed potential F is used, equation (9) becomes

$$\Delta F = 2F_x F_{xx} \tag{77}$$

If it is assumed that the boundary condition (76) may be evaluated on the axis $\omega_b = 0$ rather than on the body $\omega_b = \tau\beta\ g(x)$, then the only parameter entering the differential equation (77) and the boundary condition (76) is ϵ. The potential $F(x,\omega)$ should therefore be expressible in terms of the single parameter ϵ. On this basis, the transonic similarity rules in terms of the parameter ϵ are obtained in reference 1.

The present analysis differs from that of reference 1 in three respects. First, the boundary condition (26a), or, more generally, the boundary condition

$$r^n\varphi_r = \tau^{n+1}g^n g_x \tag{78}$$

is, in the present analysis, satisfied on the body, as

$$r_b = \tau g(x) \tag{79}$$

rather than near the axis, as

$$r \longrightarrow 0 \tag{80}$$

The use of equations (78) and (79) will evidently yield the same results for any value of the exponent n. Equation (80), however, is equivalent

to equation (79) only if $n = 1$, and then only in the first approximation; for in the first approximation (equation (41)), $\varphi \sim \log r$, so that $r\varphi_r$ has a finite nonzero limit as $r \longrightarrow 0$. Hence, reference 1 is correct insofar as the first approximation obtained herein is concerned.

In the second approximation (equation (63)), however, the dominant term is of order $\log^2 r$ so that $r\overset{2}{\varphi}_r$ becomes infinite for $r \longrightarrow 0$. This result, however, does not invalidate the procedure as regards the flow field because the boundary condition (76) is needed only to obtain the complementary solution in equation (63). As previously noted, the complementary solution is negligible with respect to the particular solution corresponding to the F_xF_{xx} term in the flow field, at least to the first three approximations.

A second difference between the present analysis and that of reference 1 stems from the logarithmic-type singularity that the potential $\varphi(x,r)$ exhibits as $r \longrightarrow 0$. This singularity affects not only the numerical value of the potential $\varphi(x,r)$ and the velocity at the body $r_b = \tau g(x)$, but also its order of magnitude. Hence, even after use of equation (79) in satisfying the boundary condition, a further use of equation (79) must be made in the solution to determine the parameters on which the velocity at the body depends.

A third difference between the present analysis and that of reference 1 is that the singularity of $\varphi(x,r)$ as $r \longrightarrow 0$ causes the terms in $\varphi_\omega\varphi_{x\omega}$ and $\varphi_\omega{}^2\varphi_{\omega\omega}$ (rather than the $\varphi_x\varphi_{xx}$ term) in the differential equation (9) to be the dominant terms in the neighborhood of the body in higher approximations. These terms lead to the similarity parameter κ in the present analysis.

The foregoing analysis indicates that there is no single transonic similarity rule for bodies of revolution for both the flow field and the body, and that the similarity rule for the body may be limited to extremely slender bodies. The use of transonic similarity for bodies of revolution may therefore be somewhat limited on a practical basis. It is well known, however, that the compressibility effects for a body of revolution are much smaller than for the corresponding two-dimensional profile (references 8 to 10), which may easily be seen from equations (43) to (45) and (63), (66), and (67) where it is noted that, on the body, the compressibility factor β occurs in the dominant terms only through powers of the slowly varying function $\log \tau\beta$. If the function $\log \tau\beta$ is considered of order 1, then the solution for $\overset{2}{\varphi}$ is of order τ^4 and should be quite small compared with the solution

for $\overset{1}{\varphi}$, which is of order τ^2. The Prandtl-Glauert rule for bodies of revolution may therefore be expected to hold for a much wider range of free-stream subsonic Mach numbers than the corresponding rule for two-dimensional bodies.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, August 31, 1950.

APPENDIX - SYMBOLS

The following symbols are used in this report:

a	local speed of sound
a_0	speed of sound in free stream
F	transformed velocity potential, $\frac{\Gamma_M}{\beta^2}\varphi$
$g(x)$	function characterizing shape of body
l	variable, λ^2-1
M_0	free-stream Mach number, U/a_0
P_n	Legendre function of first kind
Q_n	Legendre function of second kind
R	function of κ, body shape, and μ
r_n	particular integral
U	free-stream velocity
u	disturbance velocity in x-direction, φ_x
v	disturbance velocity in r-direction, φ_r
x,r,θ	cylindrical coordinates
β	compressibility factor, $\sqrt{1-M_0^2}$, (equation (4a))
Γ_M =	$M_0^2\left(1 + \frac{\gamma-1}{2} M_0^2\right)$, (equation (4b))
γ	ratio of specific heats
Δ	Laplacian
ϵ	similarity parameter in flow field, $\frac{\tau^2\Gamma_M}{\beta^2}$
κ	similarity parameter in neighborhood of body, $\tau^2 \log \tau\beta$
Λ	resultant velocity increment on body

λ,μ prolate-elliptic coordinates for body of revolution

τ lateral distance ratio

φ perturbation velocity potential

ω transformed r-coordinate, βr

Subscript:

b on the body

REFERENCES

1. von Kármán, Theodor: The Similarity Law of Transonic Flow. Jour. Math. Phys., vol. XXVI, no. 3, Oct. 1947, pp. 182-190.

2. Perl, W., and Klein, Milton M.: Theoretical Verification and Application of Transonic Similarity Law for Two-Dimensional Flow. NACA TN 2191, 1950.

3. von Kármán, Th.: Compressibility Effects in Aerodynamics. Jour. Aero. Sci., vol. 8, no. 9, July 1941, pp. 337-356.

4. Kaplan, Carl: The Flow of a Compressible Fluid past a Curved Surface. NACA Rep. 768, 1943.

5. Kaplan, Carl: Potential Flow about Elongated Bodies of Revolution. NACA Rep. 516, 1935.

6. Smythe, W. R.: Static and Dynamic Electricity. McGraw-Hill Book Co., Inc., 1939.

7. Hobson, E. W.: The Theory of Spherical and Ellipsoidal Harmonics. Univ. Press (Cambridge), 1931.

8. Sears, W. R.: A Second Note on Compressible Flows about Bodies of Revolution. Quart. Appl. Math., vol. 5, no. 1, April 1947, pp. 89-91.

9. Hess, Robert V., and Gardner, Clifford S.: Study by the Prandtl-Glauert Method of Compressibility Effects and Critical Mach Number for Ellipsoids of Various Aspect Ratios and Thickness Ratios. NACA TN 1792, 1949.

10. Lees, Lester: A Discussion of the Application of the Prandtl-Glauert Method to Subsonic Compressible Flow over a Slender Body of Revolution. NACA TN 1127, 1946.

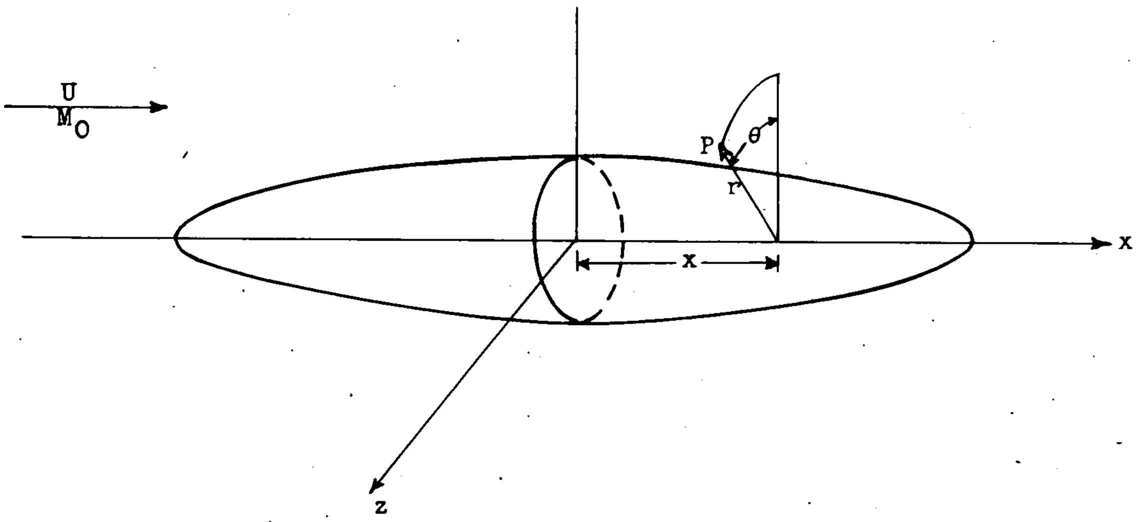

Figure 1. - Cylindrical coordinate system for body of revolution.

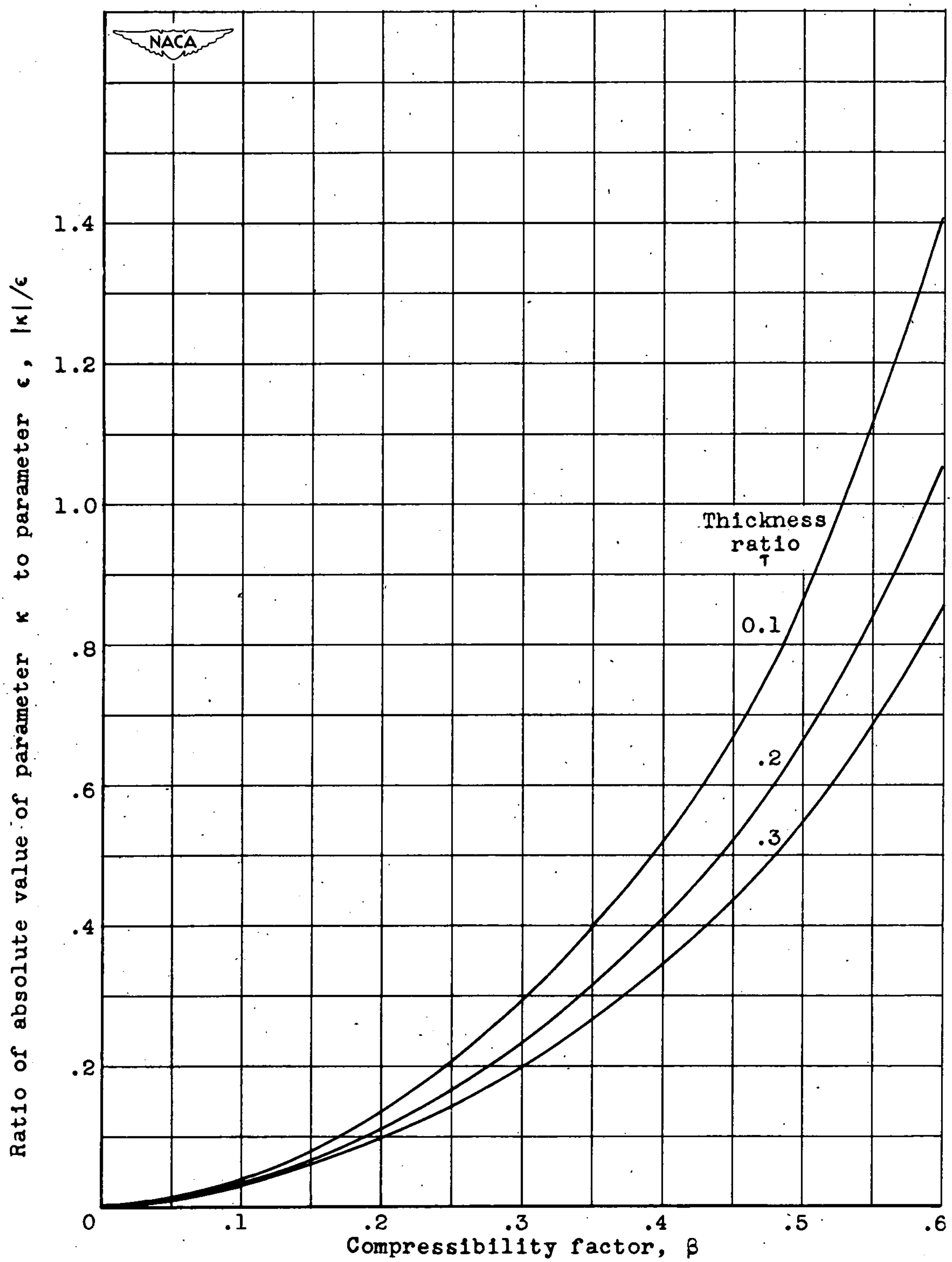

Figure 2. - Curves of ratio of absolute value of parameter κ to parameter ϵ against compressibility factor β for several thickness ratios.